

The wiggler-free free-electron laser: A single-particle model

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A general single-particle formalism is developed that is applicable for describing thermal and nonlinear effects both in the wiggler-free free-electron laser (WFFEL) and in the cyclotron autoresonance maser (CARM). The general formalism is used in the present paper for a linear analysis. The WFFEL interaction is shown to result from the coupling of the electromagnetic wave to two slow waves: a left-hand polarized wave and a longitudinal wave. This coupling results from the spatial periodicity of the electron flow in the WFFEL which follows the gyrophase coherence of the beam. The comparison between the WFFEL and the CARM is extended to the case where the wave phase velocity is different from the velocity of light in vacuum, and conditions are found for the gain in the WFFEL to be larger. An exact dispersion relation is derived for the WFFEL which employs an electron beam with thermal spread in the transverse velocity.

I. INTRODUCTION AND SUMMARY

In the wiggler-free free-electron laser (WFFEL)¹⁻⁴ a spatially periodic flow of electrons propagates along a uniform magnetic field and transfers energy to an electromagnetic wave. The mechanism of the energy transfer relies upon a coupling of the electromagnetic wave to slow waves through the periodicity of the flow, similarly to the coupling in the free-electron laser⁵ (FEL), where both the flow and the external magnetic field are spatially periodic. The spatial periodicity in the WFFEL follows the gyrophase coherence of the beam. The coupling to the slow waves does not exist in the gyrotron⁶ or in the cyclotron autoresonance maser (CARM)⁷⁻⁹ where the electron beam is randomly gyrophased and the flow is not periodic. The possible gain enhancement by the periodicity of the electron flow as a result of the gyrophase coherence of the beam was addressed also in relation to other physical systems.¹⁰⁻¹²

In this paper we progress toward an understanding of three-dimensional, thermal, and nonlinear effects. In previous papers we used fluid^{1,3} and kinetic^{2,4} pictures to describe the electron beam dynamics. Here we derive a general single-particle model, which is applicable for describing the nonlinear evolution of the electromagnetic radiation, as well as the effects of thermal spread and finite transverse system dimensions.

After laying out the general formalism in Sec. II, we turn in Sec. III to a linear analysis of the interaction of a cold beam and derive the full dispersion relation. In Sec. IV we show that the physical mechanism responsible for the WFFEL interaction relies upon coupling of the right-hand polarized electromagnetic wave to two slow waves: a left-hand polarized slow wave and a longitudinal slow wave. The contribution of the two slow waves is usually comparable. As in our previous paper⁴ we reduce the general dispersion relation to a fifth-order polynomial. At resonance four roots are nonreal, and correspondingly there are two unstable growing modes. The growth rate of the most unstable mode approximately is [Eq. (44)]

$$\text{Im } k_z = -\frac{\sin(2\pi/5)}{2^{3/5}v_z} \left[\frac{\Omega}{\gamma^3} \left(\frac{\omega_p v_t}{c} \right)^4 \right]^{1/5} \times \left[g_2 g_{-2} - 2 \left(1 - \frac{v_z}{c} \right) \left(\frac{v_z c}{v_t^2} \right) g_1 g_{-1} \right]^{1/5}.$$

Here ω_p is the beam plasma frequency, Ω is the nonrelativistic cyclotron frequency, v_t and v_z are the perpendicular and parallel components of the electron velocity, γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$, and k_z is the longitudinal component of the wave vector. The Fourier coefficients in the azimuthal gyrophase of the electron initial distribution functions $g_{\pm 1}$ and $g_{\pm 2}$ couple the right-hand polarized electromagnetic wave to a slow longitudinal wave and to a slow left-hand polarized electromagnetic wave. The term in the square brackets is usually of order one. The two terms in these brackets express the contributions of the couplings to the two slow waves. The relative roles of these two slow waves can vary according to the relative magnitudes of these two terms.

The gain in the CARM, when the transverse wave vector k_\perp is not too small, is [Eq. (33)]

$$\text{Im } k_z = -\frac{\sqrt{3}}{2} \left(\frac{\omega_p^2 v_t^2 k_\perp^2 c}{4c^2 \gamma v_z^2 \omega} \right)^{1/3},$$

where ω is the wave frequency. When k_\perp is small, the gain in the CARM is approximately [Eq. (35)]

$$\text{Im } k_z = -\frac{\omega_p}{\sqrt{2}\gamma c} \frac{v_t}{v_z}.$$

We show that in some cases of practical interest the gain in the WFFEL is significantly larger than the gain in the CARM. We show, however, that at grazing incidence the gyrophase coherence of the electron beam does not enhance the amplification.

When the density is high enough, and if the electron transverse velocity is very small, the gain in the WFFEL becomes [Eq. (50b)]

$$\text{Im } k_z = -\frac{\sqrt{3}}{2} \left(\frac{\omega_p^2 \Omega v_z^2 g_{-1}}{4c^3 \gamma^2 v_z^2 (1 + v_z/c)} \right)^{1/3}.$$

The gain scales as in the FEL in the strong-pump regime where $k_0 \equiv \Omega/\gamma v_z$ is equivalent to k_w , the wiggler wave number. We discuss the reason for this similarity. We note, however, that numerically it is usually difficult to distinguish between the two limits (48) and (54b).

In Sec. V we derive an exact dispersion relation for the WFFEL with a thermal spread in the transverse velocity. We also write the equations for the nonlinear analysis of an initially cold electron beam. This forms the basis for a future study of the crucial questions of the effects of thermal spread and nonlinearity on the interaction. We emphasize that generating the coherently gyro phased beam needed for the WFFEL without an accompanying large thermal spread has not yet been demonstrated.

In Sec. VI a numerical example is given.

II. A GENERAL NONLINEAR FORMALISM

We consider an electron beam which propagates along the z direction in the presence of both time-dependent and time-independent fields. At this stage we make the assumption that the electrons move mainly in the z direction and that their transverse excursion is small; therefore their transverse coordinates are assumed constant during their motion. Following Sprangle *et al.*,¹³ we write the general thermal distribution function f as

$$\tilde{f}(\mathbf{r}_1, z, \eta, t) = \int d\mathbf{p}_i v_{zi}(\mathbf{p}_i) \int_{-\infty}^{\infty} dt_i f(t_i, \mathbf{r}_1, \mathbf{p}_i) \delta[z - \xi(t_i, \mathbf{r}_1, \mathbf{p}_i, t)] \delta[\eta - \mathbf{p}(t_i, \mathbf{r}_1, \mathbf{p}_i, t)], \quad (1)$$

where t_i is the time the particle passes through $z=0$, \mathbf{r}_1 is the transverse coordinate of the particle at $t \geq t_i$ and \mathbf{p}_i is its momentum at $t=t_i$. The z coordinate of the particle is ξ and its momentum is \mathbf{p} . The distribution function of the particles at $z=0$ is f . The charge density $\tilde{\rho}$ and the current density \mathbf{J} , associated with this distribution function, are

$$\tilde{\rho}(\mathbf{r}_1, z, t) = -e \int d\mathbf{p}_i v_{zi}(\mathbf{p}_i) \int_{-\infty}^{\infty} dt_i f(t_i, \mathbf{r}_1, \mathbf{p}_i) \times \delta \left[\frac{t - t_i - \tau(t_i, \mathbf{r}_1, \mathbf{p}_i, z)}{|v_z(t_i, \mathbf{r}_1, \mathbf{p}_i, z)|} \right], \quad (2)$$

and

$$\mathbf{J}(\mathbf{r}_1, z, t) = -e \int d\mathbf{p}_i v_{zi}(\mathbf{p}_i) \int_{-\infty}^{\infty} dt_i f(t_i, \mathbf{r}_1, \mathbf{p}_i) \times \frac{\mathbf{p}(t_i, \mathbf{r}_1, \mathbf{p}_i, z)}{|p_z(t_i, \mathbf{r}_1, \mathbf{p}_i, z)|} \delta[t - t_i - \tau(t_i, \mathbf{r}_1, \mathbf{p}_i, z)]. \quad (3)$$

Here $-e$ is the electron charge. The time τ is the time it takes for an electron to move from the entrance to the point z ,

$$\tau(t_i, \mathbf{r}_1, \mathbf{p}_i, z) = \int_0^z \frac{dz'}{v_z(t_i, \mathbf{p}_i, \mathbf{r}_1, z')}. \quad (4)$$

We also define the flow velocity $\langle \mathbf{v} \rangle(\mathbf{r}, t) \equiv \mathbf{J}/\tilde{\rho}$. We now make the assumption¹³ that both the waves and the currents are periodic in time with frequency ω corresponding to a temporal steady state. Multiplying the Maxwell equations by $e^{i\omega t}$, and integrating over $2\pi/\omega$, we obtain

$$\left(\nabla_{\perp}^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \alpha_{\pm} - \left(\frac{\partial}{\partial x} \mp i \frac{\partial}{\partial y} \right) (\nabla \cdot \alpha) = -\frac{4\pi i \omega}{c^2} j_{\pm}, \quad (5a)$$

$$\frac{\partial \beta_y}{\partial x} - \frac{\partial \beta_x}{\partial y} + i \frac{\omega}{c} \alpha_z = \frac{4\pi}{c} j_z, \quad (5b)$$

$$\nabla \cdot \alpha = 4\pi \rho, \quad (5c)$$

and

$$\frac{\omega}{c} \beta_{\pm} = \pm \left(\frac{\partial}{\partial x} \mp i \frac{\partial}{\partial y} \right) \alpha_z \mp \frac{\partial}{\partial z} \alpha_{\pm}, \quad (5d)$$

where α , β , \mathbf{j} , and ρ are the wave electric field, the wave magnetic field, the current density, and the charge density, which are obtained by multiplying by $\exp(i\omega t)$ and by averaging over a period $2\pi/\omega$. Also $\langle \mathbf{v} \rangle(\mathbf{r}) \equiv \mathbf{j}/\rho$. Here $\alpha_{\pm} \equiv \alpha_x \mp i\alpha_y$, $\beta_{\pm} \equiv \beta_x \mp i\beta_y$, $j_{\pm} \equiv j_x \mp ij_y$. Using expressions (2)–(4) for the charge and current densities, we obtain the following expressions for \mathbf{j} and ρ :

$$\mathbf{j}(\mathbf{r}) = -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int d^3p_i v_{zi}(\mathbf{p}_i) e^{i\omega(t_i + \tau)} \times \frac{\mathbf{p}(t_i, \mathbf{p}_i, \mathbf{r}_1, \tau(t_i, \mathbf{p}_i, \mathbf{r}_1, z))}{p_z(t_i, \mathbf{p}_i, \mathbf{r}_1, \tau)} f(t_i, \mathbf{p}_i, \mathbf{r}_1), \quad (6a)$$

and

$$\rho(\mathbf{r}) = -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int d^3p_i v_{zi}(\mathbf{p}_i) e^{i\omega(t_i + \tau)} f(t_i, \mathbf{p}_i, \mathbf{r}_1) \times [v_z(t_i, \mathbf{p}_i, \mathbf{r}_1, \tau)]^{-1}. \quad (6b)$$

Here v_z and p_z were assumed to be always positive. As explained in Ref. 13 the periodicity of the currents allows us to integrate on t_i over a finite interval of length $2\pi/\omega$ only.

In order to calculate the expressions (6a) and (6b) for \mathbf{j} and ρ , we have to solve the single-particle equations of motion for the particles momentum \mathbf{p} and velocity v_z at z and average their contributions over t_i and \mathbf{p}_i . We now specify the time-independent fields to be a uniform magnetic field $\mathbf{B}_0 = B_0 \hat{e}_z$. The equations of motion are for slowly changing variables, as used, for example, by Fliflet,⁸

$$p_x + ip_y = ip_t \exp \left[i \left(\frac{\Omega}{\gamma_i} \tau + \phi \right) \right], \quad (7)$$

where p_t and ϕ are the slowly varying amplitude and phase. The nonrelativistic cyclotron frequency is $\Omega = eB_0/mc$ where m is the electron mass. The single-particle equations of motion are⁸

$$\frac{dp_t}{dt} = \text{Re}(S_t) = \text{Re} \left[i(F_x - iF_y) \exp \left[i \left(\frac{\Omega}{\gamma_i} \tau + \phi \right) \right] \right], \quad (8a)$$

$$\begin{aligned}\frac{d\phi}{dt} &= \text{Re}(S_\phi) \\ &= \text{Re}\left[-\frac{(F_x - iF_y)}{p_t} \exp\left[i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] - \frac{\Omega}{\gamma_i} \left(1 - \frac{\gamma_i}{\gamma}\right)\right],\end{aligned}\quad (8b)$$

$$\frac{dp_z}{dt} = \text{Re}(S_z) = \text{Re}(F_z), \quad (8c)$$

$$\frac{d\epsilon}{dt} = \text{Re}(S_\epsilon) = \text{Re}(-e\mathbf{v} \cdot \mathbf{E}), \quad (8d)$$

$$\mathbf{v} = c^2 \mathbf{p} / \epsilon, \quad (8e)$$

where ϵ is the total electron energy, $\gamma \equiv (1 + \mathbf{p} \cdot \mathbf{p} / m^2 c^2)^{1/2}$, and $\mathbf{F} \equiv -e[\mathbf{E} + (v/c) \times \mathbf{B}]$. We now assume that the fundamental time harmonic is dominant and neglect the higher harmonics in the equations of motion. For notational convenience we omitted the subscript 1 from the wave-field coefficients. The expressions for the sources become

$$\begin{aligned}S_t &= -\frac{ie \exp(-i\omega t)}{2} \left\{ \left(\alpha_+ - i \frac{v_z}{c} \beta_+ \right) \exp\left[i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right. \\ &\quad \left. - \left(\alpha_- + i \frac{v_z}{c} \beta_- \right) \exp\left[-i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right\},\end{aligned}\quad (9a)$$

$$\begin{aligned}S_\phi &= e \frac{\exp(i\omega t)}{2p_t} \left\{ \left(\alpha_+ - i \frac{v_z}{c} \beta_+ \right) \exp\left[i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right. \\ &\quad \left. + \left(\alpha_- + i \frac{v_z}{c} \beta_- \right) \exp\left[-i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right\} \\ &\quad + \frac{e\beta_z e^{-i\omega t}}{m_0 c \gamma} - \frac{\Omega}{\gamma_i} \left(1 - \frac{\epsilon_i}{\epsilon}\right),\end{aligned}\quad (9b)$$

$$\begin{aligned}S_z &= \frac{ev_t}{2c} \exp(-i\omega t) \left\{ \beta_+ \exp\left[i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right. \\ &\quad \left. + \beta_- \exp\left[-i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right\} - e\alpha_z e^{-i\omega t},\end{aligned}\quad (9c)$$

$$\begin{aligned}S_\epsilon &= -i \frac{ev_t}{2} \exp(-i\omega t) \left\{ \alpha_+ \exp\left[i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right. \\ &\quad \left. - \alpha_- \exp\left[-i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right)\right] \right\} - ev_z \alpha_z e^{-i\omega t}.\end{aligned}\quad (9d)$$

Using (7), we write the current density as

$$\begin{aligned}j_\pm &= \pm ie \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^\infty dp_{ti} 2\pi p_{ti} \int_{-\infty}^\infty dp_{zi} \\ &\quad \times \int_0^{2\pi} d\phi_i v_{zi} f(t_i p_{ti} p_{zi} \phi_i \mathbf{r}_1) \frac{p_t}{p_z} \\ &\quad \times \exp\left[\mp i\left(\frac{\Omega}{\gamma_i} \tau + \phi\right) + i\omega(t_i + \tau)\right],\end{aligned}\quad (10a)$$

$$\begin{aligned}j_z &= -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^\infty dp_{ti} 2\pi p_{ti} \int_0^{2\pi} d\phi_i v_{zi} \\ &\quad \times f(t_i p_{ti} p_{zi} \phi_i \mathbf{r}_1) \exp[i\omega(t_i + \tau)],\end{aligned}\quad (10b)$$

and the charge density as

$$\begin{aligned}\rho &= -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^\infty dp_{ti} 2\pi p_{ti} \int_{-\infty}^\infty dp_{zi} \int_0^{2\pi} d\phi_i v_{zi} \\ &\quad \times f(t_i p_{ti} p_{zi} \phi_i \mathbf{r}_1) \exp[i\omega(t_i + \tau)] (v_z)^{-1}.\end{aligned}\quad (10c)$$

Maxwell's equations (5) and the equations of motion (8) together with the definitions (10) fully describe the electron beam-wave interaction in very general cases. They comprise the basis for a future study of nonlinear processes, the influence of thermal spread, and three-dimensional effects.

In Eqs. (10) p_t , p_z , ϕ , and v_z are calculated at a time $t = t_i + \tau$, which is the time the particle reaches the coordinate z along its propagation. We assume that the particle velocity v_z is always positive and therefore, for each particle, τ is a single-valued function of z . The equations of motion (8) and the Maxwell equations (5) also describe the CARM interaction. Contrary to what is usually done in the CARM analysis we have retained the left-hand polarized components α_- and β_- and the longitudinal field α_z . These fields will shortly be shown to play a major role in the WFFEL interaction.

Having presented the general formalism, we derive, in the next section, the dispersion relation of the linearized equations.

III. DERIVATION OF THE FULL DISPERSION RELATION OF THE LINEARIZED EQUATIONS

We first reduce the problem to a one-dimensional (1-D) problem by assuming that the transverse dependence of the wave fields and of the beam dynamics is weak. On the left-hand side of Eq. (11a) the operator ∇_\perp^2 becomes $-k_\perp^2$. Retaining this term, we are able to study the CARM interaction when the phase velocity is different from c . Moreover, in our simplified model, this term is the only transverse-dependent term retained. We neglect the second term on the left-hand side of Eq. (5a) since this term represents the transverse derivative of $\nabla \cdot \alpha$, and $\nabla \cdot \alpha$ is proportional to the beam density, which is assumed to be low. Similarly, we neglect the transverse derivatives in Eqs. (5b) and (5d). The 1-D Maxwell equations are now

$$\left(-k_\perp^2 + \frac{d^2}{dz^2} + \frac{\omega^2}{c^2}\right) \alpha_\pm(z) = -\frac{4\pi i \omega}{c^2} j_\pm(z), \quad (11a)$$

$$i \frac{\omega}{c} \alpha_z(z) = \frac{4\pi}{c} j_z(z), \quad (11b)$$

$$\frac{d\alpha_z}{dz} = 4\pi \rho, \quad (11c)$$

and

$$\frac{\omega}{c} \beta_\pm = \mp \frac{d\alpha}{dz} \pm. \quad (11d)$$

The transverse dependence appears in our simplified analysis only in the term k_{\perp}^2 in Eq. (11a).

The electron distribution function f is assumed to be independent of \mathbf{r}_{\perp} . The 1-D Maxwell equations enable us to study the nonlinear wave-beam interaction with an arbitrary electron distribution function at $z=0$. This can be done by solving the full nonlinear single-particle equations, as was done, for example, in Ref. 13 for the FEL, or by employing a quasilinear approach within a fluid picture.¹⁴ We postpone the nonlinear analysis for the future and turn to a linear analysis.

The current and charge densities, linearized to first order in the wave fields' amplitudes, are

$$j_{\pm,1} = \pm \frac{ie\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^{\infty} dp_{ti} 2\pi p_{ti} \times \int_{-\infty}^{\infty} dp_{zi} \int_0^{2\pi} d\phi_i v_{zi} f(t_i, \mathbf{p}_i) \times \frac{p_{ti}}{p_{zi}} \exp \left[\mp i \left(\frac{\Omega}{\gamma_i} \tau_0 + \phi_i \right) + i\omega(t_i + \tau_0) \right] \times \left[\frac{p_t}{p_{ti}} - \frac{p_{z1}}{p_{zi}} \mp i \left(\frac{\Omega \tau_1}{\gamma_i} + \phi_1 \right) + i\omega \tau_1 \right], \quad (12a)$$

$$j_{z1} = -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^{\infty} dp_{ti} 2\pi p_{ti} \int_{-\infty}^{\infty} dp_{zi} \times \int_0^{2\pi} d\phi_i v_{zi} f(t_i, \mathbf{p}_i) \exp[i\omega(t_i + \tau_0)] i\omega \tau_1, \quad (12b)$$

and

$$\rho_1 = -\frac{e\omega}{2\pi} \int_0^{2\pi/\omega} dt_i \int_0^{\infty} dp_{ti} 2\pi p_{ti} \int_{-\infty}^{\infty} dp_{zi} \times \int_0^{2\pi} d\phi_i f(t_i, \mathbf{p}_i) \exp[i\omega(t_i + \tau_0)] \left(i\omega \tau_1 - \frac{v_{z1}}{v_{zi}} \right). \quad (12c)$$

To zeroth order, in the presence of the uniform magnetic field only, the quantities p_t , p_z , ϕ , and ϵ are constant and have the same values they had at $z=0$. The time τ_0 is z/v_{zi} . Despite the approximations we could still allow a general distribution function f and study thermal spread effects. Indeed, we address thermal spread effects in Sec. V. Here we limit ourselves to an initial distribution function which is cold and steady (does not depend on t_i). The electron distribution function is

$$f(t_i, p_{ti}, p_{zi}, \phi_i) = N_0 \frac{\delta(p_{ti} - p_{t0})}{2\pi p_{ti}} \delta(p_{zi} - p_{z0}) g(\phi_i). \quad (13)$$

As in our kinetic analysis,² we write

$$g(\phi_i) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} g_n e^{in\phi_i}, \quad (14)$$

and $g_n = g_{-n}$ are real, so that $g(\phi_i)$ is real. The zeroth-order current and charge densities are

$$\rho_0 = -e \int d\mathbf{p}_i \int_{-\infty}^{\infty} dt_i f \delta \left(t - t_i - \frac{z}{v_{zi}} \right) = -eN_0, \quad (15a)$$

$$j_{\pm 0} = \pm ieN_0 v_{z0} \exp(\mp ik_0 z) g_{\pm 1}, \quad (15b)$$

$$j_{z0} = -eN_0 v_{z0}, \quad (15c)$$

$$\langle \mathbf{v} \rangle_0 = \mathbf{j}_0 / \rho_0, \quad (15d)$$

where $v_0 \equiv p_0/m\gamma_0$, $\gamma_0^2 \equiv 1 + (p_{t0}^2 + p_{z0}^2)/m^2 c^2$, and

$$k_0 \equiv \Omega/(\gamma_0 v_{z0}). \quad (16)$$

We look for a solution for the fields $\alpha(z)$ and $\beta(z)$ of the form

$$\alpha_+(z) = a_+ \exp(ikz), \quad (17a)$$

$$\alpha_-(z) = a_- \exp[i(k_z + 2k_0)z], \quad (17b)$$

$$\alpha_z(z) = a_z \exp[i(k_z + k_0)z], \quad (17c)$$

$$\beta_+(z) = b_+ \exp(ik_z z), \quad (17d)$$

$$\beta_-(z) = b_- \exp[i(k_z + 2k_0)z]. \quad (17e)$$

We substitute these expressions for the fields into the sources in the single-particle equations (8). The time t is $t = t_i + \tau \equiv t_i + z/v_{zi}$. The linearized sources in the single-particle equations are, therefore,

$$S_{t1} = -\frac{ie}{2} \left[\left(a_+ - i \frac{v_{zi}}{c} b_+ \right) e^{i\phi_i} - \left(a_- + i \frac{v_{zi}}{c} b_- \right) e^{-i\phi_i} \right] e^{i\Psi}, \quad (18a)$$

$$S_{\phi 1} = \frac{e}{2p_{ti}} \left[\left(a_+ - i \frac{v_{zi}}{c} b_+ \right) e^{i\phi_i} + \left(a_- + i \frac{v_{zi}}{c} b_- \right) e^{-i\phi_i} \right] e^{i\Psi} - \frac{\Omega}{\gamma_i} \frac{\epsilon_1}{\epsilon_i}, \quad (18b)$$

$$S_{z1} = \left(\frac{v_{ti}}{2c} (b_+ e^{i\phi_i} + b_- e^{-i\phi_i}) - a_z \right) e e^{-i\Psi}, \quad (18c)$$

$$S_{\epsilon 1} = - \left(\frac{iv_{ti}}{2} (a_+ e^{i\phi_i} - a_- e^{-i\phi_i}) + v_{zi} a_z \right) e e^{i\Psi}, \quad (18d)$$

where

$$\Psi = \left(k_z + k_0 - \frac{\omega}{v_{zi}} \right) z - \omega t_i. \quad (19)$$

Expressing the linearized time derivative as $d/dt = v_{zi}(d/dz)$, we solve the linearized single-particle equations of motion and find that

$$\epsilon_1 = \text{Re} \left[-\frac{ee^{i\Psi}}{\chi} \left(\frac{v_{ti}}{2} (a_+ e^{i\phi_i} - a_- e^{-i\phi_i}) - iv_{zi} a_z \right) \right], \quad (20a)$$

$$p_{z1} = \text{Re} \left[-\frac{ee^{\Psi}}{2\chi} \left[\left(a_+ - i \frac{v_{zi}}{c} b_+ \right) e^{i\phi_i} - \left(a_- + i \frac{v_{zi}}{c} b_- \right) e^{-i\phi_i} \right] \right], \quad (20b)$$

$$\phi_1 = -\text{Re} \left[\frac{iee^{\Psi}}{2p_{ti}\chi} \left[\left(a_+ - i \frac{v_{zi}}{c} b_+ \right) e^{i\phi_i} + \left(a_- + i \frac{v_{zi}}{c} b_- \right) e^{-i\phi_i} \right] + \frac{ie\Omega e^{\Psi}}{\gamma_i \epsilon_i \chi^2} \left(\frac{v_{ti}}{2} (a_+ e^{i\phi_i} - a_- e^{-i\phi_i}) - i v_{zi} a_z \right) \right], \quad (20c)$$

$$p_{z1} = -\text{Re} \left[\frac{iee^{\Psi}}{\chi} \left(\frac{v_{ti}}{2c} (b_+ e^{i\phi_i} + b_- e^{-i\phi_i}) - a_z \right) \right], \quad (20d)$$

$$v_{z1} = \frac{p_{z1}}{m\gamma_i} - v_{zi} \frac{\epsilon_1}{\epsilon_i}, \quad (20e)$$

and

$$\begin{aligned} \tau_1 &= - \int_0^z \frac{dz' v_{z1}(z')}{v_{zi}^2} \\ &= -\text{Re} \left(\frac{iee^{\Psi}}{p_{zi}\chi^2} \left[\frac{v_{ti}}{2c} \left[\left(i b_+ - \frac{v_{zi}}{c} a_+ \right) e^{i\phi_i} + \left(i b_- + \frac{v_{zi}}{c} a_- \right) e^{-i\phi_i} \right] - i \left(1 - \frac{v_{zi}^2}{c^2} \right) a_z \right] \right), \end{aligned} \quad (20f)$$

where

$$\chi \equiv k_z v_{zi} + \frac{\Omega v_{zi}}{\gamma_0 v_{z0}} - \omega. \quad (20g)$$

We now perform the integrations over t_i , p_{ti} , p_{zi} , and ϕ_i in Eqs. (12), using expressions (20) for the linearized variables, and the form (13) of the initial distribution function. The perturbed charge density is

$$\begin{aligned} \rho_1 &= -\frac{\omega_p^2(k_z + k_0)}{4\pi\gamma_0[(\omega - k_z v_{z0} - \Omega/\gamma_0)^2 - (\omega_p^2/\gamma_0)(1 - v_{z0}^2/c^2)]} \\ &\times \left[\frac{v_{z0}}{2c} \left(\frac{k_z c}{\omega} - \frac{v_{z0}}{c} \right) a_{+g-1} + \frac{v_{z0}}{2c} \left(-\frac{(k_z + 2k_0)c}{\omega} + \frac{v_{z0}}{c} \right) a_{-g1} \right]. \end{aligned} \quad (21)$$

Here $\omega_p = (4\pi N_0 e^2/m)^{1/2}$ is the plasma frequency of the beam. Note that for beams of gyrophase coherence, where $g_{-1} \neq 0$, a_+ generates a perturbed charge density of the same form as in the conventional FEL. The coupling of the perturbed density to a_+ is proportional to the rate of gyrophase coherence g_{-1} . The perturbed current density, however, is different from that in the FEL. We write the perturbed current $j_{1,+}$ as

$$j_{1,+} \equiv \rho_0 \langle \mathbf{v} \rangle_{1,+} + \langle \mathbf{v} \rangle_{0,+} \rho_1. \quad (22)$$

In the FEL the second term is resonant and large, while the first term $\rho_0 \langle \mathbf{v} \rangle_{1,+}$ is small. In the WWFEL, however, both terms are large, and in fact, the two largest terms in the expressions for $\rho_0 \langle \mathbf{v} \rangle_{1,+}$ and $\langle \mathbf{v} \rangle_{0,+} \rho_1$ usually cancel each other out. The dominant term in the current is then a higher-order term and is smaller than in the FEL. Nevertheless, it is larger than in the CARM.

We substitute the calculated expressions for j_+ , j_- , and j_z into (11a) and (11b) and obtain three coupled algebraic equations for a_+ , a_- , and a_z with the unknown parameter (eigenvalue) k_z . These equations are

$$\begin{aligned} \left(-k_1^2 - k_z^2 + \frac{\omega^2}{c^2} \right) a_+ &= \frac{\omega_p^2}{c^2 \gamma_0} \left(\frac{(\omega - k_z v_{z0})}{(\omega - k_z v_{z0} - \Omega/\gamma_0)} a_{+g0} + \frac{v_{z0}^2 [(k_z^2 - \omega^2/c^2) a_{+g0} + (\omega^2/c^2 - k_z^2 - 2k_z k_0) a_{-g2}]}{2(\omega - k_z v_{z0} - \Omega/\gamma_0)^2} \right. \\ &\quad \left. + \frac{i v_{z0} \omega (\omega v_{z0}/c^2 - k_z)}{(\omega - k_z v_{z0} - \Omega/\gamma_0)^2} a_{zg1} \right), \end{aligned} \quad (23a)$$

$$\begin{aligned} \left(-k_1^2 - (k_z + 2k_0)^2 + \frac{\omega^2}{c^2} \right) a_- &= -\frac{\omega_p^2}{c^2 \gamma_0} \left(-\frac{(\omega - k_z v_{z0} - 2\Omega/\gamma_0)}{(\omega - k_z v_{z0} - \Omega/\gamma_0)} a_{-g0} \right. \\ &\quad \left. + \frac{v_{z0}^2 \{ (k_z^2 + 2k_z k_0 - \omega^2/c^2) a_{+g-2} + [\omega^2 - (k_z + 2k_0)^2] a_{-g0} \}}{2(\omega - k_z v_{z0} - \Omega/\gamma_0)^2} \right. \\ &\quad \left. + \frac{i v_{z0} \omega (\omega v_{z0}/c^2 - k_z - 2k_0)}{(\omega - k_z v_{z0} - \Omega/\gamma_0)^2} a_{zg-1} \right), \end{aligned} \quad (23b)$$

$$i a_z = -\frac{\omega_p^2}{\gamma_0 (\omega - k_z v_{z0} - \Omega/\gamma_0)^2} \left[\frac{v_{z0}}{2c} \left(\frac{k_z c}{\omega} - \frac{v_{z0}}{c} \right) a_{+g-1} + \frac{v_{z0}}{2c} \left(-\frac{(k_z + 2k_0)c}{\omega} + \frac{v_{z0}}{c} \right) a_{-g1} - i \left(1 - \frac{v_{z0}^2}{c^2} \right) a_{zg0} \right]. \quad (23c)$$

The special physical interaction in the wiggler-free FEL comes from the coupling of the three waves: the right-hand polarized wave α_+ (amplitude a_+), the left-hand polarized slow wave α_- (amplitude a_-), and the longitudinal slow wave α_z (amplitude a_z). The coupling of the three waves results from the gyrophase coherence of the initial distribution function. The strength of the coupling of the right-hand polarized wave and the slow left-hand polarized wave is measured by the magnitude of g_2 . The coupling between the right-hand polarized wave and the longitudinal slow wave is measured by the magnitude of g_1 . In the CARM the initial distribution function is independent of the gyrophase ϕ_i . The coefficients $g_{\pm 1}$ and $g_{\pm 2}$ are therefore zero, and the coupling of the three waves vanishes. The CARM interaction is therefore different from the WF-FEL interaction. We will discuss the differences further in the following sections.

In order to have nontrivial solutions to Eqs. (23a)–(23c) the determinant should be zero. Equating the determinant to zero, we obtain the dispersion relation, which is an equation for the eigenvalue k_z . We refer to the equation resulting from equating the determinant of (23a)–(23c) to zero as the full dispersion relation (FDR).

IV. THE ROOTS OF THE DISPERSION RELATION

We will now study the solutions of the dispersion relation and calculate the growth rates for some limiting cases, where a particular physical mechanism is dominant. To that end we apply various approximations to the FDR.

In all cases, we focus on a parameter regime where $(-k_1^2 - k_z^2 + \omega^2/c^2)$ is small, so that the wave is nearly a vacuum electromagnetic wave. We also concentrate on the case where the denominator $(\omega - k_z v_{z0} - \Omega/\gamma_0)$ is small. We write

$$\frac{\omega^2}{c^2} - k_z^2 - k_1^2 = -k_0^2(\Delta - \xi_+)(\Delta - \xi_-), \quad (24)$$

where the two mismatch parameters ξ_{\pm} , and Δ are

$$\xi_{\pm} \equiv \frac{1}{k_0} \left[k_0 - \frac{\omega}{v_{z0}} \pm \left(\frac{\omega^2}{c^2} - k_1^2 \right)^{1/2} \right], \quad (25)$$

$$\Delta \equiv \frac{1}{k_0} \left(k_z + k_0 - \frac{\omega}{v_{z0}} \right).$$

We look for cases when Δ is much smaller than 1. The spatial growth rate of the unstable modes is given by the negative imaginary part of k_z (or of $k_0\Delta$). The frequencies at which ξ_+ and ξ_- vanish correspond, respectively, to the Doppler upshifted and Doppler downshifted cyclotron frequencies. These two resonant frequencies are

$$\omega_{\pm} = \frac{\Omega/\gamma_0 \pm (v_{z0}/c) [(\Omega/\gamma_0)^2 - k_1^2 c^2 (1 - v_{z0}^2/c^2)]^{1/2}}{(1 - v_{z0}^2/c^2)}. \quad (26)$$

We discuss mainly the case of small k_1 , so that the two frequencies are distinct. We are interested in the Doppler upshifted frequency and, therefore,

$$\omega \approx \omega_0 \equiv \frac{\Omega/\gamma_0}{(1 - v_{z0}/c)}, \quad k_z \approx k_{z0} \equiv \omega_0/c. \quad (27)$$

We assume that there is a significant Doppler upshift, and that the frequency is much above the cutoff frequency, $k_{z0} \gg k_0, k_1$. Later we treat the case of grazing incidence, $\omega_+ = \omega_-$.

We turn first to the analysis of the CARM. When the beam is randomly gyrophased,

$$g(\phi_i) = 1, \quad g_{\pm 1} = 0 = g_{\pm 2}, \quad g_0 = 1, \quad (28)$$

the equations for a_+ , a_- , and a_z decouple. From Eq. (23a) we obtain

$$-k_1^2 - k_z^2 + \frac{\omega^2}{c^2} = \frac{\omega_p^2}{c^2 \gamma_0} \left(\frac{(\omega - k_z v_{z0})}{(\omega - k_z v_{z0} - \Omega/\gamma_0)} - \frac{v_{z0}^2}{2} \frac{(\omega^2/c^2 - k_z^2)}{(\omega - k_z v_{z0} - \Omega/\gamma_0)^2} \right). \quad (29)$$

The first term on the right-hand side is a stabilizing term and the second term is destabilizing. Near the Doppler upshifted resonant frequency the dispersion relation becomes

$$\left(\frac{2\omega}{ck_0} \right) (\Delta - \xi) = \frac{\omega_p^2}{k_0^2 c^2 \gamma_0} \left(\frac{1}{\Delta} + \frac{v_{z0}^2}{2v_{z0}^2} \times \frac{[-2(\omega/ck_0)(\Delta - \xi) + (k_1/k_0)^2]}{\Delta^2} \right). \quad (30)$$

This is a cubic polynomial for Δ . The scaling of the growth rate of the CARM instability depends on the dominant term in the numerator of the destabilizing term. When

$$k_1^4 \gg 2 \frac{\omega^2}{c^2} \frac{\omega_p^2}{c^2 \gamma_0} \frac{v_{z0}^2}{v_{z0}^2}, \quad (31)$$

$$\frac{2\omega_p^2}{c^2 \gamma_0} \left(\frac{\Omega}{\gamma_0 v_{z0}} \right)^3 \left(\frac{v_{z0}}{v_{z0}} \right)^4 \frac{c}{\omega}.$$

The dispersion relation is approximated as

$$\Delta^2(\Delta - \xi) = \frac{\omega_p^2 v_{z0}^2 k_1^2 c}{4k_0^2 c^2 \gamma_0 v_{z0}^2 \omega k_0}, \quad (32)$$

and the maximum growth rate, for $\xi = 0$, is

$$\text{Im } k_z = -\sqrt{\frac{3}{2}} \left(\frac{\omega_p^2 v_{z0}^2 k_1^2 c}{4c^2 \gamma_0 v_{z0}^2 \omega} \right)^{1/3}. \quad (33)$$

On the other hand, if

$$k_1^4 \ll 2 \frac{\omega^2}{c^2} \frac{\omega_p^2}{c^2 \gamma_0} \frac{v_{z0}^2}{v_{z0}^2}, \quad (34)$$

the first term in the numerator of the destabilizing term is dominant, and when the second (stabilizing) term is small, the growth rate is

$$\text{Im } k_z = -\frac{\omega_p}{\sqrt{2} \gamma_0 c} \frac{v_{z0}}{v_{z0}}. \quad (35)$$

Equations (33) and (35) therefore describe the growth rate of the instability when inequalities (31) and (34), respectively, are satisfied for the CARM interaction. We now turn to the cases where the gyrophase coherence of the electron beam plays a major role.

To simplify the analysis of the FDR, we define

$$\beta \equiv v_0/c, \quad (36a)$$

$$\eta^2 \equiv \frac{\omega_p^2}{2c^2 k_0^2 \gamma_0} \frac{\beta_t^2}{\beta_z^2}, \quad (36b)$$

$$\zeta^2 \equiv \frac{\omega_p^2}{c^2 k_0^2 \gamma_0} \frac{(1-\beta_z^2)}{\beta_z^2}, \quad (36c)$$

$$s \equiv \frac{\beta_t}{2(1+\beta_z)}, \quad (36d)$$

$$w \equiv \frac{k_1^2}{2k_0^2} \frac{(1-\beta_z)}{\beta_z}. \quad (36e)$$

We first assume that

$$\eta^2, \zeta^2 \ll \Delta^2. \quad (37)$$

In this case we approximate Eqs. (23b) and (23c) as

$$a_- = -\frac{\eta^2}{2\Delta^2} a_+ g_{-2}, \quad a_z = \frac{i\zeta^2 s}{\Delta^2} a_+ g_{-1}. \quad (38)$$

The approximated dispersion relation is

$$\Delta^2(\Delta - \xi) = \eta^2 \left((\Delta - \xi + w)g_0 + \frac{\eta^2}{2\Delta^2} g_{-2}g_2 - \frac{\zeta^2}{2\Delta^2} \frac{\beta_z}{(1+\beta_z)} g_{-1}g_1 \right). \quad (39)$$

The terms responsible for the coupling of the three waves are clearly seen. The second term on the right-hand side (rhs) of this equation represents the strength of the coupling to the slow left-hand polarized wave, while the third term represents the coupling to the slow longitudinal wave. The first term on the rhs is responsible for the CARM interaction. When the beam is randomly gyrophased, and Eqs. (28) are satisfied, only the first term remains, and the interaction is the CARM interaction. When the beam is a double helical beam,

$$g(\phi_i) = \frac{1}{2}[\delta(\phi_i) + \delta(\phi_i - \pi)], \quad g_{\pm 1} = 0, \quad g_0 = 1 = g_{\pm 2}, \quad (40)$$

the electromagnetic wave is coupled to the left-hand polarized slow wave only. There is no coupling to the longitudinal slow wave. We discussed and analyzed such a case in Ref. 4. When the beam is helical

$$g(\phi_i) = \delta(\phi_i), \quad g_n = 1, \quad (41)$$

the electromagnetic wave is coupled to both slow waves and the coupling to the longitudinal slow wave has to be considered as well.

The approximated dispersion relation is a fifth-order polynomial for Δ . When the term responsible for the CARM interaction is small, the dispersion relation becomes

$$\Delta^4(\Delta - \xi) = \frac{1}{2}\eta^4 U, \quad (42)$$

where

$$U \equiv g_2 g_{-2} - 2(1-\beta_z)(\beta_z/\beta_t^2)g_1 g_{-1}. \quad (43)$$

The maximum growth rate is

$$\begin{aligned} \text{Im } k_z &= -\sin\left(\frac{2\pi}{5}\right) k_0 \left(\frac{1}{2}\eta^4 U\right)^{1/5} \\ &= -\frac{\sin(2\pi/5)}{2^{3/5} v_z} \left[\frac{\Omega}{\gamma^3} \left(\frac{\omega_p v_t}{c}\right)^4 \right]^{1/5} \left[g_2 g_{-2} \right. \\ &\quad \left. - 2\left(1 - \frac{v_z}{c}\right) \left(\frac{v_z}{v_t}\right) g_1 g_{-1} \right]^{1/5}. \end{aligned} \quad (44)$$

We also require that the first term on the rhs of Eq. (39) be small. This is satisfied, for k_1 small, if

$$U^2 \frac{\Omega^2}{\gamma_0} \gg \frac{\omega_p^2}{2} \beta_t^2. \quad (45)$$

Note that U is usually of order one. On the other hand, $\eta^2 \ll \Delta^2$ if

$$\frac{2\omega_p^2 \beta_t^2}{c^2 k_0^2 \gamma_0 \beta_z^2 U^2} \ll 1. \quad (46a)$$

Also, for $\zeta^2 \ll \Delta^2$, and if the coupling to the longitudinal wave is at least comparable to the coupling to the left-hand polarized wave,

$$\frac{\omega_p}{c k_0 \gamma_0^{1/2} \gamma_z} \ll \frac{\beta_t^2 \beta_z^2 g_1 g_{-1}}{4(1+\beta_z)}. \quad (46b)$$

Here $\gamma_z^2 = (1-\beta_z^2)^{-1}$. Thus if the density is low enough and satisfies inequalities (46), the coupling to both waves can be comparable. The dominant coupling, either to the left-hand polarized wave, or to the space-charge wave, depends on the relative magnitudes of the two terms in U . For a helical beam, when $g_{\pm 1} = 1 = g_{\pm 2}$, both terms in square brackets in Eq. (44) are usually important. Note also that the gyrophase contribution vanishes when $U=0$.

We turn now to a second case. If

$$\eta^2 \ll \Delta^2 \ll \zeta^2, \quad (47)$$

the fields are approximated as

$$a_- = \frac{\eta^2}{2\Delta^2} \left(g_{-2} + \frac{g_{-1}^2}{2(1+\beta_z)} \right) a_+, \quad a_z = -i s a_+ g_{-1}. \quad (48)$$

Both a_- and a_z are smaller than a_+ . When we substitute these expressions for a_- and a_z into Eq. (23a) we note that the contribution of a_z is larger (because $\eta^2 \ll \Delta^2$). The dispersion relation is approximated as

$$\Delta^2(\Delta - \xi) = \eta^2 \left((\Delta - \xi + w) + \frac{\beta_z}{2(1+\beta_z)} g_1 g_{-1} \right). \quad (49)$$

The growth rate is maximal when

$$|\Delta| = \left(\frac{\eta^2 \beta_z}{2(1+\beta_z)} g_{1g-1} \right)^{1/3}, \quad (50a)$$

or

$$\text{Im } k_z = -\frac{\sqrt{3}}{2} \left(\frac{\omega_p^2 \beta_z^2 g_{1g-1} k_0}{4c^2 \gamma_0 \beta_z (1+\beta_z)} \right)^{1/3}. \quad (50b)$$

It is interesting to note that this gain scales very similarly to the gain of an FEL in the strong-pump (Compton) regime,¹⁵ when k_w , the wiggler wave number, is replaced by k_0 . However, the condition $\Delta^2 \ll \xi^2$ (47), becomes

$$\frac{\omega_p}{ck_0 \gamma_0^{1/2} \gamma_z^3} \gg \frac{\beta_z^2 \beta_z^2 g_{1g-1}}{4(1+\beta_z)}. \quad (51)$$

Note that this inequality is the reverse of inequality (46b). Inequality (51), in the case of the FEL, is typical of the operation in the collective Raman regime, contrary to operation in the strong-pump Compton regime. The source of the difference between the FEL and the wiggler-free FEL, in this respect, lies in the fact that both terms in the current in the wiggler-free FEL [Eq. (22)] are resonant. When inequality (51), or its equivalent for the FEL, is satisfied, the perturbed density ρ_1 [Eq. (21)] is not resonant. The gain in the FEL vanishes for $\xi=0$, and a different resonant frequency corresponds to the Raman regime. In the wiggler-free FEL, on the other hand, $\langle v \rangle_{1,+}$ is still resonant. The two largest terms in $\langle v \rangle_{1,+}$ and in ρ_1 that cancel each other for low density, do not do so for high density since ρ_1 is small. The gain scales then as in the FEL in the strong-pump Compton regime.

With the inequality $\eta^2 \ll \Delta^2$, the condition for the validity of (49) is

$$\frac{\beta_z^2 (1-\beta_z^2)^{3/2}}{\sqrt{2} \beta_z (1+\beta_z)} g_{1g-1} \gg \frac{\omega_p}{ck_0 \gamma_0^{1/2} \gamma_z^3} \gg \frac{\beta_z^2 \beta_z^2 g_{1g-1}}{4(1+\beta_z)}. \quad (52)$$

We note that it is very difficult to distinguish numerically between the two regimes, the high- and low-density regimes.

In a recent gyro TWA experiment¹⁶ that operated near grazing incidence [$\omega_+ = \omega_-$ in (26)], the measured wave amplification was found to be larger than the amplification predicted by the theory.⁸ A suggestion was made¹⁶ that the enhanced amplification was due to the gyrophase coherence of the beam (the WWFEL mechanism). Let us examine the operation near grazing incidence. In this case

$$k_1 \cong \gamma k_0, \quad \omega = \gamma^2 k_0 c. \quad (53)$$

The coupling coefficient to the space-charge wave in (23a) is proportional to $(\omega v_{z0} - k_z c^2)$ which vanishes at grazing incidence. The vanishing coupling to the space-charge mode has been observed recently by Chen *et al.*,¹⁷ who concluded that at grazing incidence there is no enhancement of the amplification due to the gyrophase coherence. The analysis in Ref. 17 did not take into account the coupling to the left-hand polarized wave. It is easily shown that this coupling does not vanish. However, the maximum

growth rate normalized to k_0 at grazing incidence for the CARM Γ_c and for the WWFEL Γ_w are

$$\Gamma_c = 0.87 h^{1/3}, \quad \Gamma_w = 0.82 h^{2/5}, \quad (54)$$

where

$$h \equiv \frac{\omega_p^2 v_{z0}^2}{4c^2 \gamma_0 v_{z0}^2 k_0^2} = \eta^2 / 2. \quad (55)$$

Since $h \ll 1$ the CARM growth rate is larger. Therefore, indeed, at grazing incidence the WWFEL interaction does not enhance the wave amplification.

V. THERMAL SPREAD

To zeroth order, in the presence of the uniform magnetic field only, the quantities p_\perp , p_z , ϕ , and ϵ are constant and keep their values of $z=0$. The time τ_0 is z/v_{zi} . The integration over t_i is easily performed if f is independent of t_i . Equations (5) become

$$\begin{aligned} & \left(-k_1^2 + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \alpha_+(z) \\ &= \omega \frac{4\pi}{c} e N_0 \int_0^\infty 2\pi p_{ti} dp_{ti} \int_{-\infty}^\infty dp_{zi} v_{zi} \frac{p_{ti}}{p_{zi}} \\ & \times \exp \left[i \left(\omega - \frac{\Omega}{\gamma_i} \right) \frac{z}{v_{zi}} \right] \sum_{j=0, \pm 1} \left[\frac{\tilde{p}_{ti}^{(j)}}{p_{ti}} - \frac{\tilde{p}_{zi}^{(j)}}{p_{zi}} \right] \\ & - i \left[\left(\frac{\Omega}{\gamma_i} - \omega \right) \tilde{\tau}_1^{(j)} + \tilde{\phi}_1^{(j)} \right] f_{1-j}, \end{aligned} \quad (56a)$$

$$\begin{aligned} & \left(-k_1^2 + \frac{d^2}{dz^2} + \frac{\omega^2}{c^2} \right) \alpha_-(z) \\ &= -\frac{\omega 4\pi}{c} e N_0 \int_0^\infty 2\pi p_{ti} dp_{ti} \int_{-\infty}^\infty dp_{zi} v_{zi} \frac{p_{ti}}{p_{zi}} \\ & \times \exp \left[i \left(\omega + \frac{\Omega}{\gamma_i} \right) \frac{z}{v_{zi}} \right] \sum_{j=0, \pm 1} \left[\frac{\tilde{p}_{ti}^{(j)}}{p_{ti}} - \frac{\tilde{p}_{zi}^{(j)}}{p_{zi}} \right] \\ & + i \left[\left(\frac{\Omega}{\gamma_i} + \omega \right) \tilde{\tau}_1^{(j)} + \tilde{\phi}_1^{(j)} \right] f_{-1-j}, \end{aligned} \quad (56b)$$

$$\begin{aligned} i \frac{\omega}{c} \alpha_z(z) &= -\frac{4\pi e}{c} N_0 \int_0^\infty 2\pi p_{ti} dp_{ti} \\ & \times \int_{-\infty}^\infty dp_{zi} v_{zi} i\omega \sum_{j=0, \pm 1} \tilde{\tau}_1^{(j)} f_{-j}. \end{aligned} \quad (56c)$$

Here

$$f = N_0 \sum_{n=-\infty}^\infty f_n(p_{ti} p_{zi}) e^{in\phi_i}, \quad (57)$$

the symbols with the tilde satisfy the equations

$$\frac{d\tilde{p}_{ti}^{(\pm 1)}}{dz} = \mp \frac{ie}{2} \left(\alpha_\pm \mp \frac{iv_{zi}}{c} \beta_\pm \right) \exp \left[i \left(-\omega \pm \frac{\Omega}{\gamma_i} \right) \frac{z}{v_{zi}} \right], \quad (58a)$$

$$\tilde{p}_{ti}^{(0)} = 0, \quad (58b)$$

$$\frac{d\tilde{\phi}_1^{(\pm 1)}}{dz} = \pm \frac{i}{p_{ii}} \frac{d\tilde{p}_{i1}}{dz} - \frac{\Omega}{\gamma_i \epsilon_i} \tilde{\epsilon}_1^{(\pm 1)}, \quad (58c)$$

$$\frac{d\tilde{\phi}_1^{(0)}}{dz} = -\frac{\Omega}{\gamma_i \epsilon_i} \tilde{\epsilon}_1^{(0)}, \quad (58d)$$

$$\frac{d\tilde{p}_{z1}^{(\pm 1)}}{dz} = \frac{ev_{ii}}{2c} \beta_{\pm} \exp \left[i \left(-\omega \pm \frac{\Omega}{\gamma_i} \right) \frac{z}{v_{zi}} \right], \quad (58e)$$

$$\frac{d\tilde{p}_{z1}^{(0)}}{dz} = -e\alpha_z \exp \left(\frac{i\omega z}{v_{zi}} \right), \quad (58f)$$

$$\frac{d\tilde{\tau}_1^{(j)}}{dz} = -\frac{1}{v_{zi}^2} \left(\frac{\tilde{p}_{z1}^{(j)}}{m\gamma_i} - v_{zi} \frac{\tilde{\epsilon}_1^{(j)}}{\epsilon_i} \right). \quad (58g)$$

Also, $\tilde{\epsilon}_1^{(j)}$ satisfy

$$\frac{d\tilde{\epsilon}_1^{(\pm 1)}}{dz} = \mp \frac{iev_{ii}}{2} \alpha_{\pm} \exp \left[i \left(-\omega \pm \frac{\Omega}{\gamma_i} \right) \frac{z}{v_{zi}} \right],$$

$$\frac{d\tilde{\epsilon}_1^{(0)}}{dz} = -ev_{ii}\alpha_z \exp \left(-\frac{i\omega z}{v_{zi}} \right), \quad \tilde{\epsilon}_1^{(0)} = v_{zi}\tilde{p}_{z1}^{(0)}.$$

There are two cases where these equations are simplified.

(i) No thermal spread in p_{zi} . We then write

$$f_n(p_{ii}p_{zi}) = q_n(p_{ii})\delta(p_{zi}-p_{z0}), \quad (59)$$

and integrate on p_{zi} . Then, using the definition (17) we easily perform the z integration. We obtain a system of algebraic equations for $\mathbf{a} = (a_+, a_-, a_z)$, which is a generalization of Eqs. (23). These generalized equations are

$$\mathbf{A}\mathbf{a} = \mathbf{\sigma}\mathbf{a}. \quad (60)$$

The tensor \mathbf{A} is diagonal

$$\begin{aligned} A_{++} &= -k_1^2 - k_z^2 + \omega^2/c^2, \\ A_{--} &= -k_1^2 - (k_z + 2k_0)^2 + \omega^2/c^2, \\ A_{zz} &= i. \end{aligned} \quad (61)$$

The elements of the tensor $\mathbf{\sigma}$ are

$$\begin{aligned} \sigma_{++} &= \frac{\omega_p^2}{c^2} \int_0^\infty \frac{2\pi p_{ii} dp_{ii}}{\gamma_i} \left(\frac{(\omega/v_{zi} - k_z)}{(\omega/v_{zi} - k_z - k_0)} \right. \\ &\quad \left. + \frac{p_{ii}^2}{2p_{z0}^2} \frac{(k_z^2 - \omega^2/c^2)}{(\omega/v_{zi} - k_z - k_0)^2} \right) q_0(p_{ii}), \\ \sigma_{+-} &= \frac{\omega_p^2}{c^2} \int_0^\infty \frac{2\pi p_{ii}}{\gamma_i} dp_{ii} \frac{p_{ii}^2}{2p_{z0}^2} \frac{(\omega^2/c^2 - k_z^2 - 2k_z k_0)}{(\omega/v_{zi} - k_z - k_0)^2} q_2(p_{ii}), \\ \sigma_{+z} &= i \frac{\omega_p^2}{c^2} \omega \int_0^\infty \frac{2\pi p_{ii}}{\gamma_i} dp_{ii} \frac{p_{ii}}{p_{z0}} \frac{(\omega/c^2 - k_z/v_{zi})}{(\omega/v_{zi} - k_z - k_0)^2} q_1(p_{ii}), \\ \sigma_{--} &= -\frac{\omega_p^2}{c^2} \int_0^\infty \frac{2\pi p_{ii}}{\gamma_i} dp_{ii} \left(-\frac{(\omega/v_{zi} - k_z - 2k_0)}{(\omega/v_{zi} - k_z - k_0)} \right. \\ &\quad \left. + \frac{p_{ii}^2[\omega^2/c^2 - (k_z + 2k_0)^2]}{2p_{z0}^2(\omega/v_{zi} - k_z - k_0)^2} \right) q_0(p_{ii}), \end{aligned}$$

$$\sigma_{-+} = \sigma_{+-}, \quad q_{-2} = q_2,$$

$$\begin{aligned} \sigma_{-z} &= -\frac{i\omega_p^2}{c^2} \omega \int_0^\infty \frac{2\pi p_{ii} dp_{ii}}{\gamma_i} \frac{p_{ii}}{p_{z0}} \\ &\quad \times \frac{(\omega/c^2 - k_z/v_{zi} - 2k_0/v_{zi})}{(\omega/v_{zi} - k_z - k_0)^2} q_{-1}(p_{ii}), \\ \sigma_{z+} &= -\omega_p^2 \int_0^\infty \frac{2\pi p_{ii} dp_{ii}}{\gamma_i} \frac{p_{ii}(k_z/\omega v_{zi} - 1/c^2)}{2p_{z0}(\omega/v_{zi} - k_z - k_0)^2} q_{-1}(p_{ii}), \\ \sigma_{z-} &= -\omega_p^2 \int_0^\infty \frac{2\pi p_{ii} dp_{ii}}{\gamma_i} \frac{p_{ii}[(k_z + 2k_0)/\omega v_{zi} + 1/c^2]}{2p_{z0}(\omega/v_{zi} - k_z - k_0)^2} \\ &\quad \times q_1(p_{ii}), \end{aligned}$$

$$\sigma_{zz} = i\omega_p^2 \int_0^\infty \frac{2\pi p_{ii} dp_{ii}}{\gamma_i} \frac{p_{ii}(1/v_{zi}^2 - 1/c^2)}{(\omega/v_{zi} - k_z - k_0)^2} q_0(p_{ii}). \quad (62)$$

Here $\gamma_i = (1 + p_{ii}^2/m^2 c^2 + p_{z0}^2/m^2 c^2)^{1/2}$ and $v_{zi} = p_{z0}/m\gamma_i$. These equations are reduced to Eqs. (23) if

$$q_n(p_{ii}) = \delta(p_{ii} - p_{i0})/2\pi p_{ii}. \quad (63)$$

The influence of the thermal spread in the perpendicular momentum was studied by Freund *et al.*¹¹ in connection to the auroral kilometric radiation. The equilibrium electron distribution function was chosen to be

$$\begin{aligned} f_0(p_{ii}p_{z0}\phi_i) &= (\pi\alpha_1^2)^{-1} \exp[-(p_{ii}-p_{i0})^2/\alpha_1^2] \\ &\quad \times \exp[-2p_{ii}p_{i0}(1-\cos\phi_i)/\alpha_1^2] \delta(p_z - p_{z0}). \end{aligned} \quad (64)$$

That distribution function may also be written as

$$\begin{aligned} f_0 &= (\pi\alpha_1^2)^{-1} \exp\left(-\frac{(p_{ii}^2 + p_{i0}^2)}{\alpha_1^2}\right) \delta(p_{zi} - p_{z0}) \\ &\quad \times \sum_{n=-\infty}^{\infty} I_n\left(\frac{2p_{ii}p_{i0}}{\alpha_1^2}\right) e^{in\phi_i}. \end{aligned} \quad (65)$$

Following the above analysis, the perturbed current is obtained by substituting

$$q_n(p_{ii}) = (\pi\alpha_1^2)^{-1} \exp[-(p_{ii}^2 + p_{i0}^2)/\alpha_1^2] I_n(2p_{ii}p_{i0}/\alpha_1^2) \quad (66)$$

for $n=0, \pm 1, \pm 2$. The expressions obtained by substituting Eqs. (66) into Eq. (62) are much simpler than those given in Ref. 11. Also, Eqs. (60)–(62) are exact with no need to truncate an infinite series of coupled equations.

(ii) The case in which

$$f_n(p_{ii}p_{zi}) = f_0(p_{ii}p_{z0})\delta_{n,0}. \quad (67)$$

The beam is warm but randomly gyrophased. This is the case of the CARM with a warm beam. The equations for α_+ , α_- , and α_z are then decoupled. Writing $\alpha_+(z) = a_+ \exp(ik_z z)$ in Eq. (23) we obtain the dispersion relation

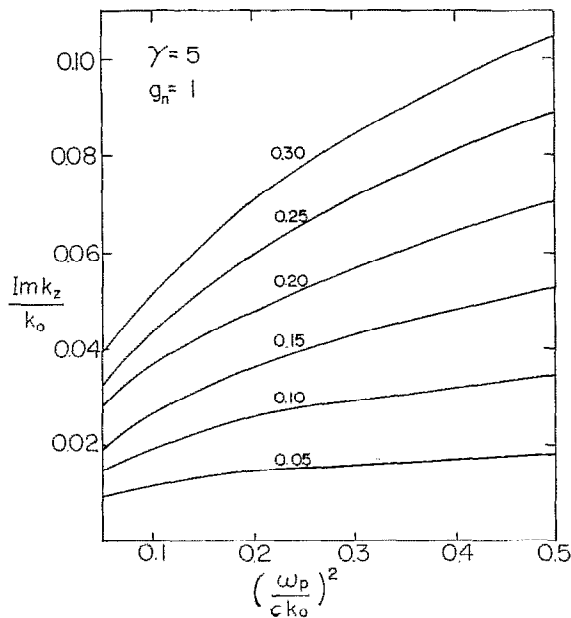


FIG. 1. The normalized growth rate ($\text{Im } k_z/k_0$) as a function of the normalized density ($\omega_p^2/c^2 k_0^2$) for various β_\perp (denoted on the curves). The beam is helical ($g_n=1$) and its energy is $\gamma=5$.

$$\begin{aligned} & \left(-k_1^2 - k_z^2 + \frac{\omega^2}{c^2} \right) \\ &= \frac{\omega_p^2}{c^2} \int_0^\infty dp_{\perp i} 2\pi p_{\perp i} \int_{-\infty}^\infty \frac{dp_{zi}}{\gamma_i} \left[\left(\frac{\omega - k_z v_{zi}}{\omega - k_z v_{zi} - \Omega/\gamma_i} \right) \right. \\ & \quad \left. + \frac{v_{\perp i}^2 (k_z^2 - \omega^2/c^2)}{2(\omega - k_z v_{zi} - \Omega/\gamma_i)^2} \right] f_0(p_{\perp i} p_{zi}). \end{aligned} \quad (68)$$

VI. NUMERICAL EXAMPLES

In this section we present numerical calculations of the amplification of the wave due to the wiggler-free FEL mechanism. Shown in Fig. 1 is the normalized growth rate ($\text{Im } k_z/k_0$) as a function of the normalized density ($\omega_p/c k_0$)², for various values of the perpendicular velocity β_\perp . The beam is helical ($g_n=1$) and its energy is $\gamma=5$. The growth rate shown is that of the most unstable mode at the frequency at which it is maximal. The growth rate was found by numerically solving the FDR [Eqs. (23)]. When $\beta_\perp=0.2$ or 0.3 , the roots of the approximated dispersion relation (39) are a good approximation. For smaller β_\perp , the approximation is less good. The normalized resonant frequencies ($\omega/c k_0$) are 13.9, 17.6, 23.0, 30.5, 38.5, and 45.6 for $\beta_\perp=0.3, 0.25, 0.2, 0.15, 0.1$, and 0.05 , respectively.

As a first example let us examine the case in which the intensity of the magnetic field is 10 kG. The cyclotron wave number k_0 is about 1.2 cm^{-1} , and hardly varies for the different β_\perp 's. The resonant wavelength varies from 3.8 mm ($\beta_\perp=0.3$) to $115 \mu\text{m}$ ($\beta_\perp=0.05$). The growth rate varies from 0.011 ($\beta_\perp=0.05$) to 0.046 cm^{-1} ($\beta_\perp=0.3$) for the lower current density (100 A/cm^2), and from 0.017 to 0.105 cm^{-1} for the higher current density (1 kA/cm^2).

As a second example let us take the magnetic field to be 50 kG. The cyclotron wave number is now 6 cm^{-1} , and the resonant wavelength varies from 760 ($\beta_\perp=0.3$) to $23 \mu\text{m}$ ($\beta_\perp=0.05$). The growth rate varies from 0.055 ($\beta_\perp=0.05$) to 0.23 cm^{-1} for the lower current density 2.5 kA/cm^2 , and from 0.085 to 0.52 cm^{-1} for the higher current density (25 kA/cm^2).

Similar to the numerical calculations in our previous publications,¹⁻⁴ the present calculations demonstrate the large amplification of the wave in the linear regime. The novelty in the present paper is the development of the general single particle formalism. The formalism developed in this paper will be the basis of a future study of the interaction in the nonlinear regime.

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- ¹A. Fruchtman and L. Friedland, J. Appl. Phys. **53**, 4011 (1982).
- ²A. Fruchtman and L. Friedland, IEEE J. Quantum Electron. **QE-19**, 327 (1983).
- ³A. Fruchtman, Phys. Fluids **29**, 1695 (1986).
- ⁴A. Fruchtman, Phys. Rev. A **37**, 4252 (1988).
- ⁵T. C. Marshall, *Free Electron Lasers* (Macmillan, New York, 1985), and references therein.
- ⁶J. L. Hirshfield, *Infrared and Millimeter Waves*, edited by K. J. Button (Academic, New York, 1979), Vol. 1.
- ⁷I. E. Botvinnik, V. L. Bratman, A. B. Volkov, G. G. Denisov, B. D. Kol'chugin, and M. M. Ofitserov, Pis'ma Zh. Tekh. Fiz. **8**, 1386 (1982) [Sov. Tech. Phys. Lett. **8**, 596 (1982)].
- ⁸A. W. Fliflet, Int. J. Electron. **61**, 1049 (1986).
- ⁹K. D. Pendergast, B. G. Danly, R. J. Temkin, and J. S. Wurtele, IEEE Trans. Plasma Sci. **PS-16**, 122 (1988).
- ¹⁰Z. G. An, Y. C. Lee, T. T. Lee, and H. H. Chen, J. Plasma Phys. **32**, 55 (1984).
- ¹¹H. P. Freund, J. Q. Dong, C. S. Wu, and L. C. Lee, Phys. Fluids **30**, 3106 (1987).
- ¹²T. H. Kho, A. T. Lin, and L. Chen, Phys. Fluids **31**, 3120 (1988).
- ¹³P. Sprangle, C. M. Tang, and W. M. Manheimer, Phys. Rev. A **21**, 302 (1980).
- ¹⁴L. Friedland (private communication, 1983).
- ¹⁵P. Sprangle, C. M. Tang, and C. W. Roberson, Nucl. Instrum. Methods A **239**, 1 (1985).
- ¹⁶S. H. Gold, D. A. Kirkpatrick, A. W. Fliflet, R. B. McCowan, A. K. Kinkad, D. L. Hardesty, and M. Suey, J. Appl. Phys. **69**, 6696 (1991).
- ¹⁷C. Chen, B. G. Danly, G. Shvets, and J. S. Wurtele, IEEE Trans. Plasma Sci. **PS-20**, 149 (1992).